# Strong Coupling Expansion of Cusp Anomalous Dimension in Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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# Outline

- What is the cusp anomalous dimension and where does it appear?
- Integrable spin chain and Bethe ansatz approach to calculate the cusp anomalous dimension
- Strong coupling expansion from BES equation
- Test of AdS/CFT correspondence

# Cusp Anomalous Dimension

Definition : Cusp anomalous dimension governs the renormalization of Wilson loops evaluated over a closed euclidean contour with a cusp [Polyakov'80]

$$\left\langle \mathrm{Tr}\ \mathrm{P}\exp\left(\imath\oint_{\mathrm{C}} dx\cdot A(x)\right)\right\rangle \sim (\Lambda_{\mathrm{UV}})^{\Gamma_{\mathrm{cusp}}(g,\vartheta)}$$

$\mathrm{C} =$ 


*Controls infrared asymptotics of scattering amplitudes in gauge theories* [Korchemsky,Radyushkin'86]

- An integration contour $\mathrm{C}$ is defined by the particle momenta
- The cusp angle $\vartheta$ is related to the scattering angles in Minkowski space

$$\Gamma_{\mathrm{cusp}}(g,\vartheta) = \vartheta\ \Gamma_{\mathrm{cusp}}(g) + \mathcal{O}(\vartheta^0) \qquad |\vartheta| \gg 1$$

Ubiquitous observable of gauge theories :

- IR singularities of on-shell gluon scattering amplitudes
- Logarithmic scaling of anomalous dimensions of high-spin Wilson operators
- ...

# Weak and Strong Coupling Expansion of Cusp in Planar $\mathcal{N} = 4$ SYM

## Cusp anomalous dimension in the 't Hooft planar limit $g^2 \equiv g_{\rm YM}^2 N_c / 16\pi^2$

- Weak coupling expansion [Kotikov,Lipatov,Onishchenko,Velizhanin'04],[Bern,Czakon,Dixon,Kosower,Smirnov'06],[Cachazo,Spradlin,Volovich'06]

$$\Gamma_{\rm cusp}(g) = 4g^2 - \frac{4}{3}\pi^2 g^4 + \frac{44}{45}\pi^4 g^6 - 8\left(\frac{73}{630}\pi^6 + 4\zeta_3^2\right) g^8 + \mathcal{O}(g^{10})$$

*Fulfills the Kotikov-Lipatov maximal transcendentality principle*

- Strong coupling expansion from AdS/CFT correspondence [Gubser,Klebanov,Polyakov'02],[Kruczenski'02],[Frolov,Tseytlin'02],[Casteil,Kristjansen'07]

$$\Gamma_{\rm cusp}(g) = 2g - \frac{3\ln 2}{2\pi} + \mathcal{O}(1/g)$$

*Semiclassical expansion of the energy of a folded string spinning in $AdS_3 \times S^1$*

- A conjecture was put forward (Beisert-Eden-Staudacher equation) about the form of the all-loop cusp anomalous dimension derived from Bethe Ansatz equations

# Anomalous Dimensions of High-Spin Operators

Wilson operators : Single-trace operators built from $L$ complex scalars $\mathcal{Z}(0)$ and $N$ lightcone derivatives $D_+$

$$\mathcal{O}_{\mathbf{n}}(0) = \mathrm{Tr}\{D_+^{n_1}\mathcal{Z}(0)...D_+^{n_L}\mathcal{Z}(0)\} \qquad \mathbf{n} = (n_1, ..., n_L) \in \mathbb{N}^L$$

Quantum numbers : Twist $L$ and Lorentz spin $N = n_1 + ... + n_L$

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim \left[(x-y)^2\right]^{-\Delta_L(N)}$$

Scaling dimensions of Wilson operators = spectrum of the dilatation operator

$$\Delta_L(N) = \Delta_L^{(0)}(N) + \delta_L(N) \qquad \delta_L(N) = \mathcal{O}(g^2)$$

Feature : Anomalous dimensions of Wilson operators $\delta_L(N)$ with large spin $N \gg 1$ occupy the band

$$2\ \Gamma_{\mathrm{cusp}}(g) \ln N \leqslant \delta_L(N) \leqslant L\ \Gamma_{\mathrm{cusp}}(g) \ln N$$

Minimal anomalous dimension has universal scaling behavior in a generic Yang-Mills theory [Korchemsky'95],[Belitsky,Gorsky,Korchemsky'03],

$$\delta_{\mathrm{min}}(N) = 2\ \Gamma_{\mathrm{cusp}}(g) \ln N + \mathcal{O}(N^0)$$

# Spin Chain Representation

## Kinematics

Single-trace Wilson composite operators built from $L$ complex scalar fields $\mathcal{Z}(0)$ and $N$ lightcone derivatives $D_+$

$$\mathcal{O}_{\mathbf{n}}(0) = \mathrm{Tr}\{D_+^{n_1}\mathcal{Z}(0)\ldots D_+^{n_L}\mathcal{Z}(0)\} \qquad \mathbf{n} = (n_1, \ldots, n_L) \in \mathbb{N}^L$$

- $\mathrm{Tr}\{\mathcal{Z}(0)\ldots\mathcal{Z}(0)\ldots\mathcal{Z}(0)\} \rightarrow$ vacuum state of the spin chain
- $\mathrm{Tr}\{\mathcal{Z}(0)\ldots D_+\mathcal{Z}(0)\ldots\mathcal{Z}(0)\} \rightarrow$ one-particle state of the spin chain (magnon)

Quantum numbers :

- Twist $L \rightarrow$ spin chain length
- Lorentz spin $N = n_1 + \ldots + n_L \rightarrow$ number of excitations (magnons) over the vacuum

## Dynamics

Wilson operators with same quantum numbers mix under a change of the renormalization scale according to the Callan-Symanzik equation

$$\mu\frac{\partial}{\partial\mu}\mathcal{O}_{\mathbf{n}}(0) = \left(\mathbb{H}\cdot\mathcal{O}\right)_{\mathbf{n}}(0) \qquad \mu\ :\ \text{renormalization scale}$$

- $\mathbb{H}$ dilatation operator of the conformal $\mathcal{N} = 4$ gauge theory $\rightarrow$Hamiltonian of the spin chain
- Spectrum of scaling dimensions $\Delta_L(N) \rightarrow$ spectrum of energies of the spin chain

# Integrability : Cusp Anomaly from Bethe Ansatz

$\mathbb{H}$ is integrable and can be solved by means of Bethe ansatz

[Lipatov'97],[Braun,Derkachov,Manashov'98],[Belitsky'98],[Korchemsky'98], [Minahan,Zarembo'02],[Beisert,Staudacher'03],[Beisert,Kristjansen,Staudacher'03],[Beisert'04]

## Bethe ansatz for dilatation operator (one-loop example)

- Bethe equations for integrable $SL(2)$ Heisenberg spin chain

$$\left(\frac{u_k + \imath/2}{u_k - \imath/2}\right)^L = \prod_{j\neq k}^{N} \frac{u_k - u_j - \imath}{u_k - u_j + \imath} \qquad \delta_L(N) = g^2 \sum_{k=1}^{N} \frac{1}{u_k^2 + 1/4}$$

Large spin limit $N \gg 1$ : continuum limit of the Bethe equations [Korchemsky'95]

- Distribution density of Bethe roots

$$\rho(z) = \frac{1}{N} \sum_{k=1}^{N} \delta\left(z - \frac{u_k}{N}\right)$$

- Bethe roots condense at the origin $\rightarrow$ logarithmic scaling $\sim \Gamma_{\text{cusp}}(g) \ln N$

  *Minimal anomalous dimensions*

$$\rho(z) = \frac{1}{\pi} \ln \frac{1+\sqrt{1-4z^2}}{1-\sqrt{1-4z^2}} \quad \rightarrow \quad \delta_{\min}(N) = g^2 \int_{-1/2}^{1/2} dz \, \frac{\rho(z)/N}{z^2 + (1/2N)^2} \sim g^2 \ln N$$

# All-Loop Asymptotic Bethe Ansatz

## Ingredients

- Exploration and understanding of integrable structures in planar $\mathcal{N} = 4$ SYM at higher loop [Serban,Staudacher'04],[Eden,Jarczak,Sokatchev'04],[Staudacher'04]
- Comparison with integrable structures on the stringy side of AdS/CFT correspondence [Bena,Polchinski,Roiban'03],[Kazakov,Marshakov,Minahan,Zarembo'04],[Beisert,Dippel,Staudacher'04]
- Necessity and presence of a dressing phase [Arutyunov,Frolov,Staudacher'04],[Beisert,Tseytlin'05],[Hernández,López'06]
- Crossing-symmetry of the dressing phase [Janik'06],[Arutyunov,Frolov'06]

## Proposal for $SL(2)$ sector

- All-loop asymptotic Bethe ansatz [Beisert,Staudacher'05],[Beisert'05]

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{N} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+}\exp\left(2\imath\theta(u_k, u_j)\right) \qquad u_k \pm \imath/2 = x_k^\pm + g^2/x_k^\pm$$

with the dressing phase $\theta(u_k, u_j)$ $\left(= \mathcal{O}\left(g^6\right)\right)$ [Beisert,Hernández,López'06],[Beisert,Eden,Staudacher'06]

- All-loop 'asymptotic' anomalous dimensions

$$\delta_L(N) = g^2 \sum_{j=1}^{N}\left[\frac{\imath}{x_j^+} - \frac{\imath}{x_j^-}\right]$$

- Wrapping effect $\rightarrow$ accuracy $= \mathcal{O}(g^{2L})$ when compared with gauge perturbation theory

# Large Spin Continuum Limit : BES equation

## BES equation for the distribution density of roots

[Eden, Staudacher'06],[Beisert, Eden, Staudacher'06]

$$\sigma(t) = \frac{t}{e^t - 1}\left(K(2gt, 0) - 4g^2 \int_0^{+\infty} dt' K(2gt, 2gt')\sigma(t')\right)$$

with the kernel

$$K(t, t') = K^{(m)}(t, t') + 2\ K^{(d)}(t, t')$$

sum of the main scattering kernel and of the dressing kernel

$$K^{(m)}(t, t') = K_0(t, t') + K_1(t, t')$$

$$K^{(d)}(t, t') = 4g^2 \int_0^{+\infty} dt'' K_1(t, 2gt'') \frac{t''}{e^{t''} - 1} K_0(2gt'', t')$$

where

$$K_0(t, t') = \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'} \sum_{n \geqslant 1} (2n-1)J_{2n-1}(t)J_{2n-1}(t')$$

$$K_1(t, t') = \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'} \sum_{n \geqslant 1} (2n)J_{2n}(t)J_{2n}(t')$$

## Relation to the cusp anomaly : $\Gamma_{\mathrm{cusp}}(g) = 8g^2\sigma(0)$

# Weak Coupling

## BES equation

$$\sigma(t) = \frac{t}{e^t - 1}\left(K(2gt,0) - 4g^2 \int_0^{+\infty} dt' K(2gt,2gt')\sigma(t')\right)$$

## Solution at weak coupling

[Eden,Staudacher'06],[Beisert,Eden,Staudacher'06]
[Belitsky'06]

$$\sigma(t) = \frac{t}{e^t - 1}\left[K(2gt,0) - 4g^2 \int_0^{\infty} dt' K(2gt,2gt')\frac{t'}{e^{t'}-1}K(2gt',0) + O(g^4)\right]$$

## Weak coupling expansion of the cusp anomaly

$$\Gamma_{\text{cusp}}(g) = 8g^2\sigma(0) = 4g^2 - \frac{4}{3}\pi^2 g^4 + \frac{44}{45}\pi^4 g^6 - 8\left(\frac{73}{630}\pi^6 + 4\zeta_3^2\right) g^8$$

$$+32\left(\frac{887}{14175}\pi^8 + \frac{4}{3}\pi^2\zeta_3^2 + 40\zeta_3\zeta_5\right) g^{10} + \mathcal{O}(g^{12})$$

- Reproduces the known four-loop result
- Verifies the KL maximal transcendentality principle to all loop [Kotikov,Lipatov'06]
- Numerical analysis indicates that the weak coupling expansion is convergent

# Strong Coupling Expansion : Numerical Approach

## Strategy

[Benna,Benvenuti,Klebanov,Scardicchio'06]

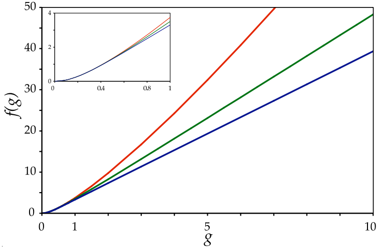
- Expand the solution $s(t) = (e^t - 1)\sigma(t)/t$ over the Bessel functions
- Truncate the series at sufficiently large number of terms $M \sim g$

$$s(t) = \sum_{n=1}^{M} s_n(g)\ \frac{J_n(2gt)}{2gt} \qquad s_{n>M}(g) = 0$$

*The integral equation becomes a finite-dimensional matrix equation for the coefficients $s_n(g)$*

- Solve numerically the matrix equation and extract the cusp anomaly $\Gamma_{\text{cusp}}(g) = 4g^2 s_1(g)$

## Results




$$f(g) = 2\Gamma_{\text{cusp}}(g) = (4.000000 \pm 0.000001)g$$
$$-(0.661907 \pm 0.000002) - \frac{0.0232 \pm 0.0001}{g} + \ldots$$

*The first two terms are in remarkable agreement with the string theory result!*

$$0.661907 = \frac{3\ln 2}{\pi}, \qquad 0.0232 = ?$$

# Strong Coupling Expansion of Cusp from BES Equation I

*Analytically, the strong coupling solution was first analyzed at leading order*

[Kotikov,Lipatov'06],[Benna,Benvenuti,Klebanov,Scardicchio'07], [Kostov,Serban,Volin'07], [Alday,Arutyunov,Benna,Eden,Klebanov'07],[Beccaria,De Angelis,Forini'07]

*and then in a more systematic approach* [B.,Korchemsky,Kotański'07],[Belitsky'07],[Kostov,Serban,Volin'08]

## Result

$$\Gamma_{\rm cusp}(g+c_1) = 2g\Big[1 - c_2 g^{-2} - c_3 g^{-3} - \left(c_4 + 2\,{c_2}^2\right) g^{-4} - \left(c_5 + 23\,c_2 c_3\right) g^{-5} - \left(c_6 + \frac{166}{7}\,c_2 c_4 + 54\,{c_3}^2 + 25\,c_2^3\right) g^{-6} + O\left(g^{-7}\right)\Big]$$

where the expansion coefficients are given by

$$c_1 = \frac{3\ln 2}{4\pi} \qquad c_2 = \frac{1}{16\pi^2}\mathrm{K} \qquad c_3 = \frac{27}{2^{11}\pi^3}\zeta(3)$$

$$c_4 = \frac{21}{2^{10}\pi^4}\beta(4) \qquad c_5 = \frac{43065}{2^{21}\pi^5}\zeta(5) \qquad c_6 = \frac{1605}{2^{15}\pi^6}\beta(6)$$

with the special functions

$$\zeta(x) = \sum_{n\geq 1} n^{-x} = \text{Riemann zeta function}$$

$$\beta(x) = \sum_{n\geq 0} (-1)^n(2n+1)^{-x} = \text{Dirichlet zeta function}$$

$$\mathrm{K} = \beta(2) = \text{Catalan's constant}$$

# Strong Coupling Expansion of Cusp from BES Equation II

## Result

$$\Gamma_{\text{cusp}}(g+c_1) = 2g\Big[1 - c_2 g^{-2} - c_3 g^{-3} - \left(c_4 + 2\,{c_2}^2\right) g^{-4}$$
$$-\left(c_5 + 23\,c_2 c_3\right) g^{-5} - \left(c_6 + \frac{166}{7}\,c_2 c_4 + 54\,{c_3}^2 + 25\,c_2^3\right) g^{-6} + O\left(g^{-7}\right)\Big]$$

## Remarkable features :

- Agreement with numerical values obtained within [Benna,Benvenuti,Klebanov,Scardicchio'07] approach
- Maximal transcendentality principle at strong coupling :

$$\begin{array}{rcl} \text{weak coupling} & \longrightarrow & \text{strong coupling} \\ \zeta(2n) & \longrightarrow & \beta(2n) \\ \zeta(2n-1) & \longrightarrow & \zeta(2n-1) \end{array}$$

- $c_1$-dependent terms inside $\Gamma_{\text{cusp}}(g)$ can be resummed by shifting $g \to g + c_1$

# AdS/CFT correspondence

$$\Gamma_{\text{cusp}}(g) = \begin{cases} \text{semiclassical energy of string spinning on AdS3} \\ \text{v.e.v. of the Wilson loop expectation value with a cusp} \end{cases}$$

[Gubser,Klebanov,Polyakov'02],[Frolov,Tseytlin'02]
[Kruczenski'02]

Perturbative string theory prediction

$$\Gamma_{\text{cusp}}(g) = 2g\left[1 - c_1 g^{-1} - c_2 g^{-2} - c_3 g^{-3} + \mathcal{O}\left(g^{-4}\right)\right]$$

with '1'= classical solution, $c_1$ = 1-loop correction, $c_2$ = 2-loop correction, ...

$c_2 = \mathrm{K}/(4\pi)^2$ was recently confirmed by superstring computation [Roiban,Tirziu,Tseytlin'07],[Roiban,Tseytlin'07]

*AdS/CFT correspondence works fine to two-loop accuracy!*

Verification of our prediction for higher order coefficients $c_k$ (with $k \leqslant 40$) remains a challenge for the string theory:

- All expansion coefficients except the first one are negative
- The expansion coefficients grow factorially at large orders

$$c_k \propto \frac{\Gamma(k-1/2)}{(2\pi)^k} \qquad \text{for } k \gg 1$$

*Strong coupling expansion of the cusp anomalous dimension is only asymptotic and is not Borel summable!*

*What is the physical meaning of these properties on the string theory side?*

# Does perturbative string theory make sense?

- Borel improved expansion of the cusp anomalous dimension

$$\Gamma_{\text{cusp}}(g) \sim -g \sum_k \frac{\Gamma(k-\frac{1}{2})}{(2\pi g)^k} = g \int_0^\infty \frac{du\, u^{-1/2} e^{-u}}{u - 2\pi g}$$

... but it is not well-defined due to a pole at $u = 2\pi g$

- The cusp anomaly (= energy of quantum spinning folded string) receives 'nonperturbative' contribution at large $g$

$$\Delta\Gamma_{\text{cusp}}(g) \sim g^{1/2} e^{-2\pi g}$$

perturbative expansion in string theory is not well-defined

*What could be an origin of such corrections?*

- Alday-Maldacena proposal:
  - Massless excitations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ sigma-model are described by noncritical $O(6)$ sigma-model with a UV cut-off set by the masses of massive (fermions + bosons) excitations
  - The $O(6)$ sigma-model develops a mass gap which affects the cusp anomalous dimension

$$\Delta\Gamma_{\text{cusp}}(g) \sim m^2 = g^{-2\beta_2/\beta_1^2} e^{4g/\beta_1}$$

Two-loop beta function for the $O(6)$ sigma-model $\beta_1 = -2/\pi$, $\beta_2 = -1/\pi^2$

# Does perturbative string theory make sense?

- Borel improved expansion of the cusp anomalous dimension

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*perfect agreement with*

Two-loop beta function for the $O(6)$ sigma-model $\beta_1 = -2/\pi$, $\beta_2 = -1/\pi^2$

# Conclusions

- BES equation can be solved analytically both at weak and strong coupling :
  - At weak coupling the cusp anomalous dimension is given by a convergent series in $g^2$
  - At strong coupling the cusp anomalous dimension is given by an asymptotic series in $1/g$
- Both at weak and strong coupling the BES interpolation agrees with the known gauge and string results
- The strong coupling expansion is non Borel-summable and suffers from non-perturbative ambiguities
- Non-perturbative corrections to the cusp anomalous dimension are governed by the dynamical infrared scale of the $O(6)$ sigma-model
- Quantitative description of these non-perturbative corrections is still missing